

HOPF ALGEBRA STRUCTURE OF GENERALIZED QUASI-SYMMETRIC FUNCTIONS IN PARTIALLY COMMUTATIVE VARIABLES

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ABSTRACT. We introduce a generalization of the Hopf algebra of quasi-symmetric functions in terms of power series in partially commutative variables. This is the graded dual of the Hopf algebra of coloured non-commutative symmetric functions described as a subalgebra of the Hopf algebra of rooted ordered coloured trees. In the Appendix we discuss the role of partial commutativity in derivation of Weyl commutation relations.

1. INTRODUCTION

Theory of Hopf algebras [1, 46] forms a modern basis for understanding symmetries of solvable models in quantum and statistical theoretical physics [36, 31, 8]. Application of Hopf algebras to combinatorics can be traced back to Rota [29], see also [43, 23] for more recent reviews of the subject, which has expanded since then. Hopf algebras of trees appeared in the analysis of Runge–Kutta methods by Butcher [6] and Dür [18], and in works by Grossman and Larson [24] in the context of symbolic computation. More recently they were used by Connes and Kreimer [10] to describe renormalization procedure of quantum field theory, see also [3, 4]. The non-commutative Hopf algebra of trees and forests, generalizing that of Connes and Kreimer, was considered by Foissy [19], and independently by Holtkamp [28].

The theory of symmetric functions [45, 35] is by now well established subject with numerous applications in algebraic topology, combinatorics, representation theory, integrable systems and geometry. Quasi-symmetric functions, introduced by Gessel [22] (see also an earlier relevant work of Stanley [44]), are extensions of symmetric functions that are becoming of comparable importance. As a graded Hopf algebra, the dual of the algebra of quasi-symmetric functions is the Hopf algebra of non-commutative symmetric functions introduced by Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon [21].

In works of Zhao [51] and Hoffman [27] there was established isomorphism between the Hopf algebra of non-commutative symmetric functions and certain subalgebra of rooted ordered trees Hopf algebra. One of aims of our work is to generalize this connection. We construct first the Hopf algebra NSym_n of n -coloured non-commutative symmetric functions starting from the Hopf algebra of n -coloured rooted ordered trees by Foissy. Then we define n -coloured quasi-symmetric functions QSym_n as graded dual of NSym_n providing also its realization in terms of certain power series of bounded degree in partially commuting variables. Partially commutative variables have been introduced to study combinatorial problems by Cartier and Foata in [7]. They also have found applications in algebra, theory of orthogonal polynomials, statistical physics and computer science; see review by Viennot [50] written in terms of heaps of pieces. In theoretical computer science, as was proposed by Mazurkiewicz [37], they describe concurrent computations.

Relation between integrable systems, both classical and quantum, and algebraic combinatorics appears in multifold aspects. Already the standard description of the Kadomtsev–Petviashvili (KP) hierarchy of integrable partial differential equations in terms of free fermions by the Kyoto School [39] involves large part of the theory of symmetric functions, see also a generalization [13] in direction of quasi-symmetric functions. Often the τ -functions of integrable hierarchies are related to known generating functions for various combinatorial objects [25]. Combinatorial aspects of the Bethe ansatz and of the quantum inverse scattering method [32] were studied, for example, in works by Fomin, Kirillov and Reshetikhin [20, 30]. For more about mutual interactions between the theory of integrable systems and combinatorics, see recent reviews [12, 52].

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Another developing area of applications of integrable systems close to combinatorics is the theory of random partitions and random matrices, see reviews [11, 49, 2], where the famous Tracy–Widom distribution [48] is constructed using a solution of the Painlevé II equation. The family of Painlevé equations, both differential and difference, has astonishing description in terms of the affine Weyl groups, as given by Sakai [42]. Other combinatorial faces of the Painlevé equations and their relation to the discrete KP (or the Hirota–Miwa [26, 38]) equation, known in combinatorics as the octahedron recurrence [33], are presented in [41]. The affine Weyl group structure of symmetries of the non-commutative discrete KP equation by Nimmo [40] was given in our paper [14].

Results presented in this article are intended as a step to build a connection, on the basis of algebraic combinatorics, between the renormalization procedure and the theory of integrable systems. The paper is organized as follows. In Section 2 we recall first relevant notions and results on Hopf algebras of trees, and then we present necessary background from the theory of symmetric, quasi-symmetric and non-commutative symmetric functions. In the next Section 3 we define the algebra of n -coloured non-commutative symmetric functions and we investigate its Hopf structure. Then we study its graded dual, which in the monochromatic $n = 1$ case reduces to the standard quasi-symmetric functions. We also describe such n -coloured quasi-symmetric functions in terms of power series in partially commutative variables. In the Appendix we provide an additional result on connection between partial commutativity relations and the Weyl commutation relations, which was one of our motivations for the research.

We remark that in recent book [34] one can find a brief history of quasi-symmetric functions together with a review of their various generalizations. We were not able to identify none of them with the generalization presented in our work.

2. HOPF ALGEBRAS OF TREES AND QUASI-SYMMETRIC FUNCTIONS

We assume that the Reader is familiar with the basic definitions and properties of Hopf algebras, as covered in [1] or [46]. All the results presented in this Section are known, but we recall them to provide necessary terminology and background to formulate new ones in the next Section. In the paper all algebras are over fixed field \mathbb{k} of characteristic zero, although sometimes a commutative ring may be enough.

2.1. Hopf algebras. By $(\mathcal{H}, \mu, \eta, \Delta, \epsilon)$ denote a bialgebra which is:

- (1) an associative algebra (\mathcal{H}, μ, η) consisting of \mathbb{k} -linear multiplication $\mu: \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ and \mathbb{k} -linear unit map $\eta: \mathbb{k} \rightarrow \mathcal{H}$ satisfying properties described by the diagrams:

$$(2.1) \quad \begin{array}{ccc} \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\text{id} \otimes \mu} & \mathcal{H} \otimes \mathcal{H} \\ \mu \otimes \text{id} \downarrow & & \downarrow \mu \\ \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\mu} & \mathcal{H} \end{array} \quad \begin{array}{ccc} \mathcal{H} \otimes \mathbb{k} & \xlongequal{\quad} & \mathcal{H} \xlongequal{\quad} \mathbb{k} \otimes \mathcal{H} \\ \text{id} \otimes \eta \downarrow & & \text{id} \downarrow \quad \downarrow \eta \otimes \text{id} \\ \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\mu} \mathcal{H} & \xleftarrow{\mu} \mathcal{H} \otimes \mathcal{H} \end{array}$$

- (2) a co-associative coalgebra $(\mathcal{H}, \Delta, \epsilon)$ consisting of \mathbb{k} -linear comultiplication $\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ and \mathbb{k} -linear counit map $\epsilon: \mathcal{H} \rightarrow \mathbb{k}$ satisfying properties described by the diagrams:

$$(2.2) \quad \begin{array}{ccc} \mathcal{H} & \xrightarrow{\Delta} & \mathcal{H} \otimes \mathcal{H} \\ \Delta \downarrow & & \downarrow \Delta \otimes \text{id} \\ \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\text{id} \otimes \Delta} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \end{array} \quad \begin{array}{ccc} \mathcal{H} \otimes \mathcal{H} & \xleftarrow{\Delta} \mathcal{H} & \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \\ \text{id} \otimes \epsilon \downarrow & & \text{id} \downarrow \quad \downarrow \epsilon \otimes \text{id} \\ \mathcal{H} \otimes \mathbb{k} & \xlongequal{\quad} \mathcal{H} \xlongequal{\quad} \mathbb{k} \otimes \mathcal{H} \end{array}$$

- (3) such that $\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ and $\epsilon: \mathcal{H} \rightarrow \mathbb{k}$ are unital algebra morphisms.

Bialgebra \mathcal{H} is *graded* if it is graded as \mathbb{k} -module $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}^{(n)}$ with the structure maps respecting the gradation

$$(2.3) \quad \mathcal{H}^{(n)} \otimes \mathcal{H}^{(m)} \xrightarrow{\mu} \mathcal{H}^{(n+m)}, \quad \mathcal{H}^{(n)} \xrightarrow{\Delta} \bigoplus_{n'+n''=n} \mathcal{H}^{(n')} \otimes \mathcal{H}^{(n'')}.$$

A graded bialgebra is *connected* if $\mathcal{H}^{(0)} \cong \mathbb{k}$.

The space of \mathbb{k} -linear operators $\text{End}(\mathcal{H})$ can be equipped with the *convolution product* $\star: \text{End}(\mathcal{H}) \otimes \text{End}(\mathcal{H}) \rightarrow \text{End}(\mathcal{H})$ defined for $f, g \in \text{End}(\mathcal{A})$ as follows

$$(2.4) \quad f \star g = \mu \circ (f \otimes g) \circ \Delta.$$

Such a product is associative with neutral element $\eta \circ \epsilon$. A bialgebra \mathcal{H} is called a *Hopf algebra* if there is an element $S \in \text{End}_{\mathbb{k}}(\mathcal{H})$, called *antipode*, which is two-sided inverse under \star for the identity map $\text{id}_{\mathcal{H}}$, which means

$$(2.5) \quad \text{id}_{\mathcal{H}} \star S = S \star \text{id}_{\mathcal{H}} = \eta \circ \epsilon.$$

When it exists, the antipode S is unique and is algebra anti-endomorphism: $S(1) = 1$, and $S(ab) = S(b)S(a)$ for all $a, b \in \mathcal{H}$. In this paper we will not be concerned by existence of the antipode, because all bialgebras we consider are graded and connected.

Theorem 2.1 ([47]). *Any graded connected bialgebra is a Hopf algebra.*

Two Hopf \mathbb{k} -algebras \mathcal{A}, \mathcal{B} are *dually paired* by a map $\langle \cdot, \cdot \rangle: \mathcal{B} \otimes \mathcal{A} \rightarrow \mathbb{k}$ if

$$(2.6) \quad \langle b_1 b_2, a \rangle = \langle b_1 \otimes b_2, \Delta_{\mathcal{A}}(a) \rangle, \quad \langle 1_{\mathcal{B}}, a \rangle = \epsilon_{\mathcal{A}}(a),$$

$$(2.7) \quad \langle \Delta_{\mathcal{B}}(b), a_1 \otimes a_2 \rangle = \langle b, a_1 a_2 \rangle, \quad \epsilon_{\mathcal{B}}(b) = \langle b, 1_{\mathcal{A}} \rangle$$

$$(2.8) \quad \langle S_{\mathcal{B}}(b), a \rangle = \langle b, S_{\mathcal{A}}(a) \rangle$$

which is then extended to tensor products pairwise. This means that the product of \mathcal{A} and coproduct of \mathcal{B} are adjoint to each other under $\langle \cdot, \cdot \rangle$, and vice-versa. Likewise, the units and counits are mutually adjoint, and the antipodes are adjoint. When the Hopf algebra \mathcal{H} is finite dimensional then the natural pairing between the \mathbb{k} -module \mathcal{H} and its dual \mathcal{H}^* allows to introduce on the latter the dual Hopf algebra structure.

When \mathcal{H} is infinite dimensional then there is no such general construction, which is caused by the fact that the inclusion $\mathcal{H}^* \otimes \mathcal{H}^* \subset (\mathcal{H} \otimes \mathcal{H})^*$ fails to be equality. For connected graded Hopf algebra $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}^{(n)}$ which is locally finite (each homogeneous component $\mathcal{H}^{(n)}$ is finite dimensional), one can define its *graded dual* as $\mathcal{H}^{gr} = \bigoplus_{n \geq 0} \mathcal{H}^{(n)*}$ which has the property that $\mathcal{H}^{gr} \otimes \mathcal{H}^{gr} = (\mathcal{H} \otimes \mathcal{H})^{gr}$ and $(\mathcal{H}^{gr})^{gr} \cong \mathcal{H}$. Then $\mathcal{H}^{gr} \subset \mathcal{H}^*$ is a Hopf algebra where the evaluation map $\mathcal{H}^{gr} \otimes \mathcal{H} \rightarrow \mathbb{k}$ provides a duality pairing of \mathcal{H} with \mathcal{H}^{gr} .

2.2. Hopf algebra structures on rooted ordered coloured trees. A rooted ordered tree (called also rooted plane tree) is a finite rooted tree t such that for each vertex v of t , the children of v are totally ordered (from left to right on our pictures). Such an ordering induces a natural linear order on the vertex set $V(t)$ of the tree obtained from left-to-right depth-first search; see Figure 1. By the trivial rooted tree we understand the tree consisting of the root only. A planted rooted tree is a non-trivial rooted tree, such that its root has only one child.

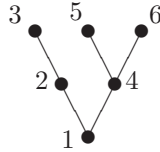


FIGURE 1. A rooted ordered tree with induced natural linear order on the vertex set

A rooted ordered coloured (ROC) tree is a rooted tree t together with a function from a set $E(t)$ of its edges to the set $\{1, 2, \dots, n\}$ of n colours. By $\mathbb{k}T_n$ denote the linear space of finite formal combinations of n -coloured rooted ordered trees with coefficients in the field \mathbb{k} . The space $\mathbb{k}T_n$ is graded with the weight $|t|$ of a ROC-tree t being the number of its edges

$$(2.9) \quad \mathbb{k}T_n = \bigoplus_{k \geq 0} \mathbb{k}T_n^{(k)}.$$

By the well known connection [45] between rooted ordered trees and Catalan numbers C_k , dimension of each graded component $\mathbb{k}T_n^{(k)}$ equals

$$(2.10) \quad \dim T_n^{(k)} = n^k C_k = \frac{n^k}{k+1} \binom{2k}{k}.$$

Define the product "·" on $\mathbb{k}T_n$ as the concatenation of trees by identification of their roots; see Figure 2 for an example. The product respects the gradation

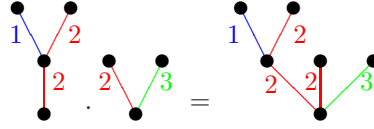


FIGURE 2. Multiplication of two coloured ordered rooted trees

$$(2.11) \quad \mathbb{k}T_n^{(k_1)} \cdot \mathbb{k}T_n^{(k_2)} \subset \mathbb{k}T_n^{(k_1+k_2)},$$

is associative with the trivial tree \bullet being the neutral element (i.e. the unit map $\eta: \mathbb{k} \rightarrow \mathbb{k}T_n$ is defined by $1 \mapsto \bullet$).

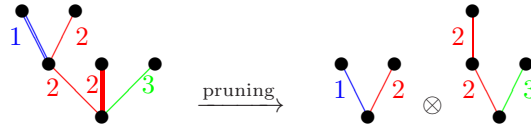


FIGURE 3. Pruning of a ROC-tree; pruned branches are thickened

In order to define compatible coproduct on $\mathbb{k}T_n$ one has first to describe the operation of pruning of a tree. A rooted subtree t_s of a ROC-tree t is called *admissible* if it shares the root of t . Such an admissible subtree is again ROC-tree with the root, order and colours inherited from t . The set of admissible subtrees of t (including the trivial tree and t itself) will be denoted by $A(t)$. Given such admissible subtree $t_s \subset t$ it defines a sequence (t_1, \dots, t_m) of planted trees being branches of t pruned to get t_s , with the order in the sequence inherited from the order on t . By concatenation of the pruned branches we obtain the *complementary tree* $t_c = t_1 \cdots t_m$ to the admissible subtree t_s of t . Such a pruning operation gives an element $t_c \otimes t_s$; see Figure 3 for an example.

The coproduct of a tree is defined as sum of pairs $t_c \otimes t_s$ for all admissible subtrees of t ; see Figure 4

$$(2.12) \quad \Delta(t) = \sum_{t_s \in A(t)} t_c \otimes t_s,$$

and then extended to $\mathbb{k}T_n$ by linearity.

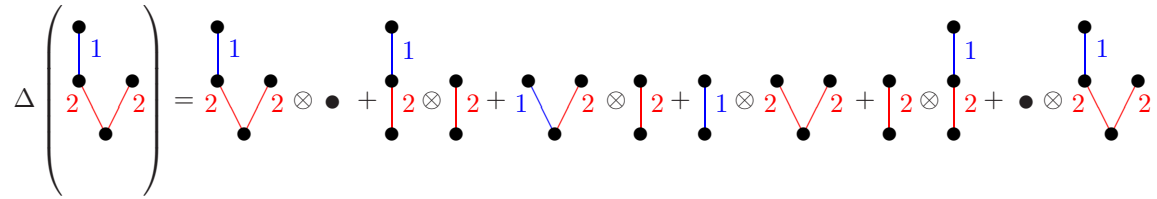


FIGURE 4. The pruning coproduct of a ROC-tree

Such coproduct is coassociative, respects the gradation

$$(2.13) \quad \Delta(\mathbb{k}T_n^{(k)}) \subset \bigoplus_{k_1+k_2=k} \mathbb{k}T_n^{(k_1)} \otimes \mathbb{k}T_n^{(k_2)},$$

and is compatible with the counit defined on trees as

$$(2.14) \quad \epsilon(t) = \begin{cases} 1 & \text{if } t = \bullet, \\ 0 & \text{otherwise.} \end{cases}$$

In this context it is convenient to define the operation B_i^+ of planting a tree on a new root by attaching it to the old one by additional edge coloured by i . In particular, planting allows to define the coproduct recursively starting from $\Delta(\bullet) = \bullet \otimes \bullet$, and using then the formula

$$(2.15) \quad \Delta(B_i^+(t)) = B_i^+(t) \otimes \bullet + (\text{id} \otimes B_i^+) \circ \Delta(t),$$

together with compatibility of the coproduct Δ with the concatenation product. Equation (2.15) has the following simple meaning: apart from the trivial subtree, all admissible subtrees of a planted tree contain the lowest (i.e. incident to the root) edge.

Theorem 2.2 ([19]). *The concatenation multiplication and pruning coproduct with the corresponding unit and counit maps equip $\mathbb{k}T_n$ with the structure of graded locally finite and connected bialgebra (thus Hopf algebra).*

Remark. Actually, Foissy works in an equivalent setting of rooted ordered vertex-coloured (or decorated) forests. Any ROC tree is uniquely mapped, by deletion of the root, to an ordered forest colouring first its vertices using colours of adjacent edges below them; see the bijection map visualized on Figure 5.

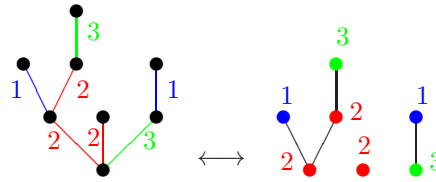


FIGURE 5. Transition from the setting of ROC- trees to the setting of ROD-forests

Because the natural basis of the finite dimensional subspace $\mathbb{k}T_n^{(k)}$ is provided by ROC-trees of weight k it seems natural to represent the dual basis of $(\mathbb{k}T_n^{(k)})^*$ by such trees again, i.e. functional $\phi_t \in (\mathbb{k}T_n^{(k)})^*$ defined on trees by

$$(2.16) \quad \langle \phi_t, t' \rangle = \begin{cases} 1 & \text{if } t = t', \\ 0 & \text{otherwise} \end{cases}$$

by standard abuse of notation is identified with t . The dual (deconcatenation) product $\delta = (\cdot)^*$ to the concatenation product acts on trees as

$$(2.17) \quad \delta(t) = \sum_{t', t'' \in T_n} \langle \phi_t, t' \cdot t'' \rangle t' \otimes t'' = \sum_{t', t'' = t} t' \otimes t'',$$

i.e. when $t = t_1 \dots t_m$ is planted trees decomposition, then

$$(2.18) \quad \delta(t_1 \dots t_m) = \sum_{i=0}^m (t_1 \dots t_i) \otimes (t_{i+1} \dots t_m),$$

see Figure 6.

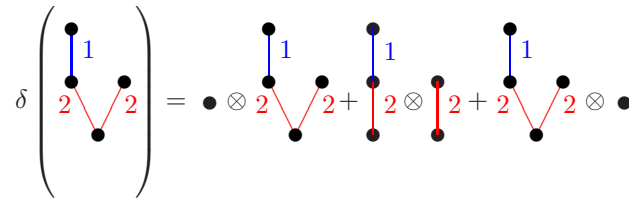


FIGURE 6. The deconcatenation coproduct of a ROC-tree

The product $\sqcup = \Delta^*$, dual to the pruning coproduct satisfies

$$(2.19) \quad \Delta(t) = \sum_{t', t'' \in T_n} \langle \phi_t, t' \sqcup t'' \rangle t' \otimes t'',$$

and is defined with the help of the grafting procedure that follows from comparison of equations (2.12) and (2.19). Given ROC-tree $t = t_1 \dots, t_m$ decomposed into the planted factors, its grafting on ROC-tree t' is defined as attaching roots of the factors t_i to vertices of t' in a way, which preserves the original ordering of the factors. In other words, a grafting of t on t' gives a tree \tilde{t} such that there exists a pruning with $t' = \tilde{t}_s$ with the corresponding $t = \tilde{t}_c$; see Figure 7 for an example. With the grafting product \lrcorner , \lrcorner ,

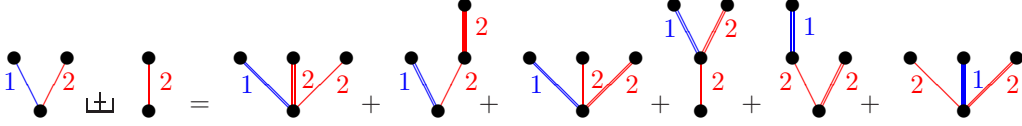


FIGURE 7. The asymmetric shuffle (or grafting) product of two ROC-trees; grafted branches are thickened

deconcatenation coproduct δ , the unit $\eta = \epsilon^*$ and counit $\epsilon = \eta^*$ maps, the space spanned by ROC-trees is equipped with another bialgebra (thus Hopf algebra) structure – the graded dual to the previous one.

It is therefore remarkable fact, discovered by Foissy, that the duality described above is self-duality. The situation is analogous to the well known self-duality of the Hopf algebra of symmetric functions (see below). The Foissy result follows from existence of a special inner product (\cdot, \cdot) on $\mathbb{k}T_n$ which respects the gradation.

2.3. The Hopf algebra of symmetric functions. Let $x = \{x_1, x_2, x_3, \dots\}$ denote infinite set of commuting variables, and let $\mathbb{k}[[x]] = \mathbb{k}[[x_1, x_2, x_3, \dots]]$ be the algebra of formal power series consisting of series of bounded degree, where each x_i has degree one. Such a formal series is called *symmetric function* if the coefficient at any term $x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$ with i_1, i_2, \dots, i_k *distinct*, agrees with that at $x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition, i.e. weakly decreasing finite sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1$. Then the *monomial symmetric function* m_λ is defined by

$$(2.20) \quad m_\lambda = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \dots x_{i_k}^{\lambda_k},$$

where the sum is over all k -tuples (i_1, i_2, \dots, i_k) of distinct indices that yield distinct monomials; by definition $m_\emptyset = 1$. The monomial symmetric functions form a linear basis of the graded algebra Sym of symmetric functions, each graded component $\text{Sym}^{(n)}$ is spanned by those m_λ for which $\lambda \vdash n$ is a partition of $n = \lambda_1 + \lambda_2 + \dots + \lambda_k$.

It is convenient to define the n -th *complete homogeneous symmetric function*

$$(2.21) \quad h_n = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \dots x_{i_n},$$

with the convention that $h_0 = 1$. It is known that they generate the algebra Sym of symmetric functions

$$(2.22) \quad \text{Sym} = \mathbb{k}[h_1, h_2, h_3, \dots].$$

Given partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, then the *complete homogeneous symmetric function* h_λ is defined by

$$(2.23) \quad h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_k}.$$

From their definitions it is clear that

$$(2.24) \quad h_\lambda h_\mu = h_{\widetilde{\lambda \cdot \mu}},$$

where $\widetilde{\lambda \cdot \mu}$ is the partition obtained by taking the multiset union of partitions λ and μ , and then reordering the union to make its elements weakly decreasing.

There is well known (graded, locally finite, and connected) Hopf algebra structure on Sym with compatible (with above product) comultiplication defined with the help of the "variables doubling". We introduce additional system of variables $y = \{y_1, y_2, \dots\}$, and to obtain the coproduct $\Delta(f)$ of a

symmetric function f we expand the function over the doubled variables, decompose resulting expression into sum of products of functions of x and y getting this way

$$(2.25) \quad f \mapsto f(x) \mapsto f(x, y) = \sum_j f'_j(x) f''_j(y) \mapsto \sum_j f'_j \otimes f''_j = \Delta(f).$$

Coassociativity follows from uniqueness of the decomposition

$$(2.26) \quad f(x, y, z) = \sum_j f'_j(x) f''_j(y) f'''_j(z).$$

In particular, one has

$$(2.27) \quad \Delta(h_n) = \sum_{i=0}^n h_i \otimes h_{n-i}, \quad \text{and} \quad \Delta(m_\lambda) = \sum_{\lambda=\mu.\nu} m_\mu \otimes m_\nu.$$

Example 2.1. Applying the procedure to $m_{(2,1)} = x_1^2 x_2 + x_2^2 x_1 + \dots$ we have

$$\begin{aligned} x_1^2 x_2 + x_2^2 x_1 + \dots &\mapsto x_1^2 x_2 + x_2^2 x_1 + \dots + x_1^2 y_1 + \dots + y_1^2 x_1 + \dots + y_1^2 y_2 + y_2^2 y_1 + \dots = \\ &= m_{(2,1)}(x) + m_{(2)}(x) m_{(1)}(y) + m_{(1)}(x) m_{(2)}(y) + m_{(2,1)}(y) \end{aligned}$$

getting this way

$$(2.28) \quad \Delta(m_{(2,1)}) = m_{(2,1)} \otimes 1 + m_{(2)} \otimes m_{(1)} + m_{(1)} \otimes m_{(2)} + 1 \otimes m_{(2,1)}.$$

The counit of the Hopf algebra of symmetric functions in the basis m_λ is given by

$$(2.29) \quad \epsilon(m_\lambda) = \begin{cases} 1 & \text{if } \lambda = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The coproduct formula in the basis of monomial homogeneous functions (2.27) is dual to the product formula in the basis of complete homogeneous symmetric functions. This gives self-duality of the Hopf algebra of symmetric functions. The proof makes use of the *Hall inner product* in Sym , for which the basis of monomial homogeneous symmetric functions is dual to the basis of complete homogeneous symmetric functions

$$(2.30) \quad (m_\pi, h_\mu) = \delta_{\pi\mu}.$$

2.4. Hopf algebra of quasi-symmetric functions and its graded dual. Let $x = \{x_1, x_2, x_3, \dots\}$ denote infinite *totally ordered* set of commuting variables, and let $\mathbb{k}[[x]] = \mathbb{k}[[x_1, x_2, x_3, \dots]]$ be again the algebra of formal power series consisting of series of bounded degree. Such a formal series is called *quasi-symmetric function* if the coefficient of any term $x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_k}^{n_k}$ with $i_1 < i_2 < \dots < i_k$ *strictly increasing*, agrees with that of $x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$. The linear space QSym of quasi-symmetric functions has as a basis the *monomial quasi-symmetric functions* indexed by compositions (finite sequences of positive integers) $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$

$$(2.31) \quad M_\alpha = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k},$$

where the sum is over all k -tuples (i_1, i_2, \dots, i_k) of strictly increasing; by definition $M_\emptyset = 1$. The algebra QSym is graded with each graded component $\text{QSym}^{(m)}$ spanned by those M_α for which $\alpha \models m$ is a composition of $m = \alpha_1 + \alpha_2 + \dots + \alpha_k$. By the well known bijection [34] any such composition can be identified with a subset $\{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \dots + \alpha_{k-1}\}$ of $m-1$, therefore $\dim \text{QSym}^{(m)} = 2^{m-1}$.

The algebra of symmetric functions Sym is a subalgebra of QSym , and for a partition $\lambda \vdash n$ and all compositions $\alpha \models n$, which give λ after reordering, we have

$$(2.32) \quad m_\lambda = \sum_{\tilde{\alpha}=\lambda} M_\alpha.$$

The product in QSym , inherited from the standard multiplication of power series, can be described in the basis (M_α) in terms of the quasi-shuffles \sqcup of compositions: in addition to shuffling components α_i

and β_j of two compositions $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_l)$ we may replace any number of pairs of consecutive components α_i and β_j in the shuffle by their sum $\alpha_i + \beta_j$

$$(2.33) \quad M_\alpha M_\beta = \sum_{\gamma=\alpha \sqcup \beta} M_\gamma.$$

Example 2.2. For $M_{(1)} = x_1 + x_2 + \dots$ and $M_{(2)} = x_1^2 + x_2^2 + \dots$ we have

$$(x_1 + x_2 + \dots)(x_1^2 + x_2^2 + \dots) = (x_1 x_2^2 + x_1 x_3^2 + \dots) + (x_1^2 x_2 + x_1^2 x_3 + \dots) + (x_1^3 + x_2^3 + \dots),$$

therefore we obtain

$$(2.34) \quad M_{(1)} M_{(2)} = M_{(1,2)} + M_{(2,1)} + M_{(3)}.$$

The coproduct δ in the algebra of quasi-symmetric functions can be defined again using the doubling variables trick. Here to the totally ordered set of variables $x = \{x_1, x_2, x_3, \dots\}$ we add its copy $y = \{y_1, y_2, y_3, \dots\}$ placing elements of y *after* elements of x , and getting the ordered sum of the sets of variables. In the basis of monomial quasi-symmetric functions (M_α) the coproduct formula reads

$$(2.35) \quad \delta(M_\alpha) = \sum_{\beta \cdot \gamma = \alpha} M_\beta \otimes M_\gamma,$$

where $\beta \cdot \gamma$ is concatenation of two compositions. As a result we obtain graded, locally finite and connected bialgebra (thus Hopf algebra) which is commutative but not cocommutative.

Example 2.3. Applying the procedure to $M_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + \dots$ we have

$$\begin{aligned} x_1^2 x_2 + x_1^2 x_3 + \dots &\mapsto x_1^2 x_2 + x_1^2 x_3 + \dots + x_1^2 y_1 + x_1^2 y_2 + \dots + y_1^2 y_2 + y_1^2 y_3 + \dots = \\ &= M_{(2,1)}(x) + M_{(2)}(x) M_{(1)}(y) + M_{(2,1)}(y) \end{aligned}$$

getting this way

$$(2.36) \quad \delta(M_{(2,1)}) = M_{(2,1)} \otimes 1 + M_{(2)} \otimes M_{(1)} + 1 \otimes M_{(2,1)}.$$

The graded dual to QSym is called the Hopf algebra of non-commutative symmetric functions [21] and denoted by NSym. Let (H_α) be the dual basis to (M_β)

$$(2.37) \quad \langle H_\alpha, M_\beta \rangle = \delta_{\alpha\beta},$$

then by dualization of equations (2.33) and (2.35) we obtain the product and coproduct formulas in NSym

$$(2.38) \quad H_\alpha H_\beta = H_{\alpha \cdot \beta}, \quad \Delta(H_\alpha) = \sum_{\beta \sqcup \gamma = \alpha} H_\beta \otimes H_\gamma.$$

In particular, for a composition $\alpha = (\alpha_1, \dots, \alpha_k)$ one has

$$(2.39) \quad H_\alpha = H_{\alpha_1} \dots H_{\alpha_k},$$

where we wrote $H_j = H_{(j)}$ for a composition (j) . The dual element to $M_\emptyset = 1$ is $H_\emptyset = H_0 = 1$. This leads to the conclusion that, as algebra, NSym is freely generated by non-commuting elements H_1, H_2, \dots

$$(2.40) \quad \text{NSym} = \mathbb{k}\langle H_1, H_2, \dots \rangle,$$

compare with equation (2.22) for the commutative case. The coproduct formula for the generators follows from equations (2.38) and reads

$$(2.41) \quad \Delta(H_m) = \sum_{j=0}^m H_j \otimes H_{m-j},$$

compare with equations (2.27).

It is known [51, 27] that the Hopf algebra NSym is isomorphic to a Hopf subalgebra of rooted ordered (monochromatic) trees generated by ladders $\ell_m = (B^+)^m(\bullet)$, where in the monochromatic $n = 1$ case we skip the lower index $i = 1$ describing the colour of attached edge by the operator B_i^+ . The isomorphism is given simply by $H_m \leftrightarrow \ell_m$. An element of the basis of the subalgebra is concatenation of several ladders, and can be called a *bunch of ladders*. The pruning coproduct of a bunch decomposes into bunches, in particular the pruning coproduct of a ladder gives a direct counterpart of the formula (2.41).

3. THE HOPF ALGEBRA OF COLOURED NON-COMMUTATIVE SYMMETRIC FUNCTIONS AND ITS DUAL

We define below n -coloured analogue of algebra of bunches of ladders as a subalgebra of ROC trees. It will turn out that it is indeed a Hopf algebra generalizing NSym. Then we study its graded dual, which has therefore dual Hopf algebra structure. Finally, we establish isomorphism of this dual to a subalgebra of power series of bounded degree in variables which satisfy partial commutation relations of Cartier–Foata type.

3.1. Coloured non-commutative symmetric functions.

Definition 3.1. Given finite sequence $I = (i_1, i_2, \dots, i_l)$ of elements from the set of colours $\{1, 2, \dots, n\}$, consider a ROC tree of the form

$$(3.1) \quad \mathbf{H}_I = (B_{i_l}^+ \circ \dots \circ B_{i_1}^+)(\bullet),$$

which can be called a *flower* (or a coloured ladder if we would like to follow nomenclature of the monochromatic case). A ROC-tree $\mathbf{H}_{I_1} \cdot \mathbf{H}_{I_2} \dots \mathbf{H}_{I_k}$ will be called a *bunch of flowers* and denoted by $\mathbf{H}_{(I_1, I_2, \dots, I_k)}$ or $\mathbf{H}_{\mathcal{I}}$ where $\mathcal{I} = (I_1, I_2, \dots, I_k)$, is a multisequence. The subalgebra of $\mathbb{k}T_n$ generated by coloured ladders will be called the algebra of n -coloured non-commutative symmetric functions and denoted by NSym_n .

Proposition 3.1. *Bunches of flowers form linear basis of NSym_n with the products*

$$(3.2) \quad \mathbf{H}_{\mathcal{I}} \cdot \mathbf{H}_{\mathcal{J}} = \mathbf{H}_{\mathcal{I} \sqcup \mathcal{J}},$$

where $\mathcal{I} \sqcup \mathcal{J}$ is juxtaposition of multisequences \mathcal{I} and \mathcal{J} .

Remark. In some papers by NSym_n it is denoted a graded component of NSym , which in our paper is $\text{NSym}^{(n)}$.

Corollary 3.2. *On the level of basis $(\mathbf{H}_{\mathcal{I}})$ of NSym_n indexed by multisequences, in the transition to the monochromatic case of $\text{NSym} = \text{NSym}_1$ instead of the sequence I_j (of ones) being a part of \mathcal{I} , we give its length $|I_j|$. Then, instead of the multisequences $\mathcal{I} = (I_1, I_2, \dots, I_k)$, elements of the basis (H_{α}) are indexed by the corresponding compositions $\alpha = (|I_1|, |I_2|, \dots, |I_k|)$. In the next step from non-commutative symmetric functions NSym to (commutative) symmetric functions Sym the corresponding linear basis (h_{λ}) is indexed by partitions obtained from compositions by reordering, i.e. $\lambda = \tilde{\alpha}$.*

Remark. Instead of multisequence $\mathcal{I} = (I_1, I_2, \dots, I_k)$ one can use corresponding *tableaux*, being left-justified array of k rows of lengths $|I_1|, |I_2|, \dots, |I_k|$, each row filled with the corresponding sequence.

Proposition 3.3. *The algebra NSym_n is a Hopf subalgebra of $\mathbb{k}T_n$ with the concatenation product and pruning coproduct.*

Proof. Concatenation of two bunches gives (almost by definition) a bunch of flowers. Also pruning of a bunch gives a splitting into bunches, in particular the pruning coproduct of a flower reads

$$(3.3) \quad \Delta(\mathbf{H}_{(i_1, i_2, \dots, i_k)}) = \sum_{j=0}^k \mathbf{H}_{(i_1, \dots, i_j)} \otimes \mathbf{H}_{(i_{j+1}, \dots, i_k)}.$$

All other properties needed to make NSym_n a Hopf algebra follow from the above and the Hopf algebra structure of $\mathbb{k}T_n$. \square

Corollary 3.4. *The Hopf algebra NSym_n is graded, locally finite and connected. The weight of $\mathbf{H}_{\mathcal{I}}$ is $|I_1| + |I_2| + \dots + |I_k|$, where $\mathcal{I} = (I_1, I_2, \dots, I_k)$. By colouring the monochromatic bunches it is easy to see that for $m \geq 1$ dimension of the graded component is $\dim \text{NSym}_n^{(m)} = n^m 2^{m-1}$.*

Remark. For $n > 1$ the coproduct in NSym_n is not cocommutative, for example

$$(3.4) \quad \Delta(\mathbf{H}_{(1,2)}) = \bullet \otimes \mathbf{H}_{(1,2)} + \mathbf{H}_{(1)} \otimes \mathbf{H}_{(2)} + \mathbf{H}_{(1,2)} \otimes \bullet.$$

An example of the pruning coproduct acting on a bunch of flowers is actually visualized on Figure 4 which in our notation can be obtained from

$$(3.5) \quad \Delta(\mathbf{H}_{(1,2),(2)}) = (\bullet \otimes \mathbf{H}_{(1,2)} + \mathbf{H}_{(1)} \otimes \mathbf{H}_{(2)} + \mathbf{H}_{(1,2)} \otimes \bullet) \cdot (\bullet \otimes \mathbf{H}_{(2)} + \mathbf{H}_{(2)} \otimes \bullet).$$

3.2. Graded dual of \mathbf{NSym}_n . Consider the graded dual $(\mathbf{NSym}_n)^{gr}$ of the Hopf algebra of n -coloured non-commutative symmetric functions, by $(\mathbf{H}_{\mathcal{I}}^*)$ denote (temporarily) the dual basis to the basis $(\mathbf{H}_{\mathcal{J}})$ of bunches in \mathbf{NSym}_n

$$(3.6) \quad \langle \mathbf{H}_{\mathcal{I}}^*, \mathbf{H}_{\mathcal{J}} \rangle = \delta_{\mathcal{I}\mathcal{J}}.$$

Abusing again the notation we represent each $\mathbf{H}_{\mathcal{I}}^*$ by a bunch of flowers, but we introduce new multiplication and coproduct (because the unit and counit maps are mutually adjoint, effectively they remain the same) that make by duality $(\mathbf{NSym}_n)^{gr}$ a Hopf algebra.

The dual to the concatenation product is the deconcatenation coproduct δ , which by dualization of equation (3.2) is given by

$$(3.7) \quad \begin{aligned} \delta(\mathbf{H}_{\mathcal{I}}^*) &= \sum_{\mathcal{I}=\mathcal{I}'\sqcup\mathcal{I}''} \mathbf{H}_{\mathcal{I}'}^* \otimes \mathbf{H}_{\mathcal{I}''}^* \quad \text{i.e.} \\ \delta(\mathbf{H}_{(I_1, I_2, \dots, I_k)}^*) &= \sum_{j=0}^k \mathbf{H}_{(I_1, I_2, \dots, I_j)}^* \otimes \mathbf{H}_{(I_{j+1}, \dots, I_k)}^* . \end{aligned}$$

In particular, flowers are primitive elements of the coproduct.

The product in $(\mathbf{NSym}_n)^{gr}$ can be defined directly by dualization of the pruning coproduct of bunches. Equivalently, it can be described in terms of the original grafting product of trees and the dual to the injection map $\mathbf{NSym}_n \hookrightarrow \mathbf{k}T_n$. It is therefore restriction of the grafting product from ROC-trees to bunches, i.e. we can graft flowers at the root of a bunch or on the top of another flower, with the restriction that two flowers cannot be grafted on the same top.

The geometric procedure on the level of trees is the same as that in the monochromatic case, so we keep the same name and symbol of the quasi-shuffle product. Quasi-shuffle of two multisequences $\mathcal{I} = (I_1, I_2, \dots, I_k)$ and $\mathcal{J} = (J_1, J_2, \dots, J_m)$ is thus a shuffle of components I_i and J_j of \mathcal{I} and \mathcal{J} , where in addition we may replace any number of pairs of consecutive components I_i, J_j in the shuffle by the concatenation $I_i J_j$ of the sequences. Therefore we have

$$(3.8) \quad \mathbf{H}_{\mathcal{I}}^* \sqcup \mathbf{H}_{\mathcal{J}}^* = \sum_{\mathcal{K}} \mathbf{H}_{\mathcal{K}}^*,$$

where the sum is over all quasi-shuffles of the multisequences \mathcal{I} and \mathcal{J} .

Example 3.1. The quasi-shuffle product of two bunches, whose asymmetric shuffle product is given in Figure 7, takes the form

$$(3.9) \quad \mathbf{H}_{(1),(2)}^* \sqcup \mathbf{H}_{(2)}^* = 2\mathbf{H}_{(1),(2),(2)}^* + \mathbf{H}_{(1),(2,2)}^* + \mathbf{H}_{(1,2),(2)}^* + \mathbf{H}_{(2),(1),(2)}^* ,$$

compare also with Figure 8.

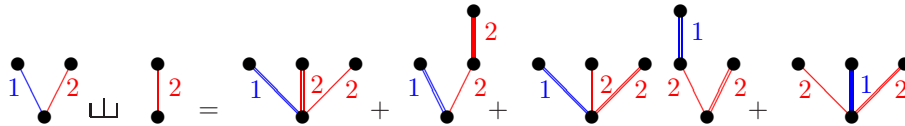


FIGURE 8. The quasi-shuffle product of two bunches as given in equation (3.9); grafted flowers are thickened

Remark. Actually, one can consider free \mathbf{k} -module generated by multisequences (or corresponding tableaux) and equip it with two dual graded Hopf structures with concatenation product and pruning coproduct, or quasi-shuffle product and deconcatenation coproduct. The unit (counit) in both structures is given by the (coefficient at the) empty multisequence.

3.3. Generalized quasi-symmetric functions. For each colour index $a \in \{1, 2, \dots, n\}$, let $x_a = \{x_{a,1}, x_{a,2}, x_{a,3}, \dots\}$ denote infinite totally ordered set of variables, which *partially commute*

$$(3.10) \quad x_{a,i}x_{b,j} = x_{b,j}x_{a,i} \quad \text{for} \quad i \neq j, \quad 1 \leq a, b \leq n,$$

i.e. variables with the second index different commute. The second index defines also partial order relation between variables of different colours.

Remark. In quantum physics, requirement of the form "elements with different indices commute" is referred often as the *ultralocality condition*.

Corollary 3.5. *Notice that within each set (the first index is fixed) all variables commute. In particular, for $n = 1$ we obtain totally ordered set of commuting variables.*

Consider the algebra $\mathbb{k}[[x_1, x_2, \dots, x_n]] = \mathbb{k}[[x_{1,1}, x_{1,2}, \dots, x_{2,1}, x_{2,2}, \dots, x_{n,1}, x_{n,2}, \dots]]$ of formal power series of bounded degree in n sets of variables $x_{a,i}$, $a = 1, 2, \dots, n$, $i = 1, 2, \dots$. Due to partial commutativity any monomial can be uniquely ordered such that second indices of variables in the monomial form weakly increasing (finite) sequence, say $i_1 \leq i_2 \leq \dots \leq i_k$. Given such monomial X , by $I_{i_j}(X)$, $j = 1, \dots, k$, denote the non-empty sequence of the first indices with the second one being j . Then the multisequence $\mathcal{I} = (I_{i_1}, I_{i_2}, \dots, I_{i_l})$ determine the ordered monomial, which will be denoted by $X_{\mathcal{I}}$. Two ordered monomials $X_{I_{i_1}, I_{i_2}, \dots, I_{i_k}}$ and $X_{I_{j_1}, I_{j_2}, \dots, I_{j_l}}$ are called *similar* if $k = l$ and $I_{i_1} = I_{j_1}, \dots, I_{i_k} = I_{j_l}$. This is indeed an equivalence relation.

Example 3.2. Let $k = 2$ and $i_1 = 1, i_2 = 3$ with the sequences $I_1 = (2, 1, 1)$ and $I_3 = (1, 2)$, then

$$(3.11) \quad X_{I_1, I_3} = x_{2,1}x_{1,1}^2x_{1,3}x_{2,3}.$$

It is similar to the monomial $x_{2,i}x_{1,i}^2x_{1,j}x_{2,j}$ for arbitrary $1 \leq i < j$.

Definition 3.2. A formal series in $\mathbb{k}[[x_1, x_2, \dots, x_n]]$ is called *n -coloured (generalized) quasi-symmetric function* if the coefficient of any monomial agrees with that of any similar monomial. The linear space of such functions is denoted by QSym_n .

Remark. In some papers by QSym_n it is denoted a graded component of QSym , which in our paper is $\text{QSym}^{(n)}$.

Any such quasi-symmetric function is therefore determined by coefficients at monomials X_{I_1, I_2, \dots, I_k} .

Definition 3.3. Given finite multisequence $\mathcal{I} = (I_1, I_2, \dots, I_k)$ with I_i , $i = 1, \dots, k$ being a finite sequence with elements from the set $\{1, 2, \dots, n\}$, by $M_{\mathcal{I}}$ denote *monomial n -coloured quasi-symmetric function* of the form

$$(3.12) \quad M_{I_1, I_2, \dots, I_l} = X_{I_1, I_2, \dots, I_l} + \text{"all other similar monomials"}.$$

By definition, for the empty multisequence we have $M_{\emptyset} = 1$.

Example 3.3. The function containing the monomials considered in Example 3.2 is

$$(3.13) \quad M_{(2,1,1),(1,2)} = x_{2,1}x_{1,1}^2x_{1,2}x_{2,2} + x_{2,1}x_{1,1}^2x_{1,3}x_{2,3} + x_{2,2}x_{1,2}^2x_{1,3}x_{2,3} + \dots$$

Proposition 3.6. *Functions $M_{\mathcal{I}}$ for all possible finite multisequences $\mathcal{I} = (I_1, I_2, \dots, I_k)$ form a linear basis of n -coloured quasi-symmetric functions.*

Following ideas from the theory of quasi-symmetric functions ($n = 1$) we define in QSym_n the structure of Hopf algebra. Multiplication in the algebra QSym_n is inherited from multiplication of formal power series subject to Cartier-Foata partial commutation relations (3.10). In particular, product of two elements of the linear basis, after reordering to have second index weakly increasing, can be described in terms of quasi-shuffles of multisequences.

Proposition 3.7. *Product of two quasi-symmetric functions from the monomial basis reads*

$$(3.14) \quad M_{\mathcal{I}}M_{\mathcal{J}} = \sum_{\mathcal{K}} M_{\mathcal{K}},$$

where the sum is over all quasi-shuffles \mathcal{K} of the multisequences \mathcal{I} and \mathcal{J} .

Proof. In the product of two monomial functions M_{I_1, I_2, \dots, I_k} and M_{J_1, J_2, \dots, J_l} consider a product of two particular ordered monomials

$$X_{I_{i_1}, I_{i_2}, \dots, I_{i_k}} X_{J_{j_1}, J_{j_2}, \dots, J_{j_l}}.$$

If all indices of the two multisequences are different then after reordering we obtain new monomial indexed by a shuffle of the components. If there is a pair of coinciding indices, say $i_p = j_q$, then in the reordered monomial the corresponding sequences will be concatenated to $I_{i_p} J_{j_q}$. \square

Example 3.4. The product of two quasi-symmetric functions

$$M_{(1),(2)} = x_{1,1}x_{2,2} + x_{1,1}x_{2,3} + x_{1,2}x_{2,3} + \dots \quad \text{and} \quad M_{(2)} = x_{2,1} + x_{2,2} + x_{2,3} + \dots,$$

reads

$$\begin{aligned} M_{(1),(2)} M_{(2)} &= (x_{1,1}x_{2,2} + x_{1,1}x_{2,3} + x_{1,2}x_{2,3} + \dots)(x_{2,1} + x_{2,2} + x_{2,3} + \dots) = \\ &= (x_{1,1}x_{2,1}x_{2,2} + \dots) + (x_{1,1}x_{2,2}x_{2,2} + \dots) + 2(x_{1,1}x_{2,2}x_{2,3} + \dots) + (x_{2,1}x_{1,2}x_{2,3} + \dots) = \\ &= M_{(1,2),(2)} + M_{(1),(2,2)} + 2M_{(1),(2),(2)} + M_{(2),(1),(2)}. \end{aligned}$$

compare with Example 3.1 and Figure 8.

We can see therefore that the product in QSym_n is described in the same way as the quasi-shuffle product in $(\text{NSym}_n)^{gr}$ under the identification

$$(3.15) \quad H_{\mathcal{I}}^* \leftrightarrow M_{\mathcal{I}}.$$

of their bases. It turns out that in transition to power series realization in terms of generalized quasi-symmetric functions also the deconcatenation coproduct in $(\text{NSym}_n)^{gr}$ allows for the variables doubling interpretation.

Let us add to the initial totally ordered sets of variables $x_a = \{x_{a,1}, x_{a,2}, \dots\}$, $a = 1, 2, \dots, n$, their copies $y_a = \{y_{a,1}, y_{a,2}, \dots\}$, $a = 1, 2, \dots, n$, whose elements satisfy partial commutation relations (3.10). With respect to second indices we place the new variables after the old ones (independently of the first index), which in particular implies that the new and old variables commute. In order to define coproduct in QSym_n we extend the notion of similar monomials to include the new variables.

Example 3.5. New similar monomials to those considered in Example 3.2 are

$$\begin{aligned} x_{2,i}x_{1,i}^2y_{1,j}y_{2,j} & \quad \text{for arbitrary } 1 \leq i, j, \\ y_{2,i}y_{1,i}^2y_{1,j}y_{2,j} & \quad \text{for arbitrary } 1 \leq i < j. \end{aligned}$$

The variables doubling of n -coloured quasi-symmetric functions, where we add all *similar* monomials involving old and new variables, is defined by equation (2.25) in the same way as for symmetric functions.

Example 3.6. Applying the procedure to $M_{(1,2),(2)} = x_{1,1}x_{2,1}x_{2,2} + \dots$ we have

$$\begin{aligned} x_{1,1}x_{2,1}x_{2,2} + \dots &\mapsto x_{1,1}x_{2,1}x_{2,2} + \dots + x_{1,1}x_{2,1}y_{2,1} + \dots + y_{1,1}y_{2,1}y_{2,2} + \dots = \\ &= M_{(1,2),(2)}(x) + M_{(1,2)}(x)M_{(2)}(y) + M_{(1,2),(2)}(y), \end{aligned}$$

getting this way

$$\delta(M_{(1,2),(2)}) = M_{(1,2),(2)} \otimes 1 + M_{(1,2)} \otimes M_{(2)} + 1 \otimes M_{(1,2),(2)}.$$

Proposition 3.8. *The variables doubling method gives for elements $M_{\mathcal{I}}$ of the monomial basis of QSym_n under identification (3.15) the same deconcatenation coproduct as in $(\text{NSym}_n)^{gr}$*

$$(3.16) \quad \delta(M_{(I_1, I_2, \dots, I_k)}) = \sum_{j=0}^k M_{(I_1, I_2, \dots, I_j)} \otimes M_{(I_{j+1}, \dots, I_k)}.$$

Proof. It is enough to consider how the variables doubling trick works for the monomial $X_{(I_1, I_2, \dots, I_k)}$. Finiteness of the multisequences assures finite sum decomposition. \square

We have therefore obtained realization of the graded Hopf dual $(\text{NSym}_n)^{gr}$ to the Hopf algebra of n -coloured non-commutative symmetric functions in terms of power series in partially commutative variables.

Remark. There is no need to check coassociativity of the coproduct in QSym_n or its compatibility with the (quasi-shuffle) product, because this holds by definition of $(\text{NSym}_n)^{gr}$.

Theorem 3.9. *Generalized n -coloured quasi-symmetric functions QSym_n with the natural product of power series, natural unit and counit maps, and coproduct given by variables doubling form graded (by length of monomials), locally finite, and connected Hopf algebra. The monomial basis elements $M_{\mathcal{I}} = M_{(I_1, I_2, \dots, I_k)}$ of $\text{QSym}_n^{(m)}$ are indexed by multisequences of length $m = |I_1| + |I_2| + \dots + |I_k|$.*

4. CONCLUSION

We have defined n -coloured generalization of the Hopf algebra of quasi-symmetric functions providing also its realization in terms of power series of bounded degree in partially commuting variables. Our definition of the Hopf algebra structure of n -coloured non-commutative symmetric functions NSym_n exploited simple observation that the product and coproduct in the Hopf algebra of n -coloured rooted ordered trees by Foissy allowed is stable on bunches of flowers. This gave for free associativity, coassociativity of the product and coproduct, and their compatibility. Then, also for free, we have got the Hopf algebra structure in the graded dual of NSym_n , which we identified with the n -coloured generalized quasi-symmetric functions QSym_n .

In the literature there are known several generalizations of the quasi-symmetric functions. We strongly believe that the generalization proposed in our paper will be useful in studying problems from combinatorics and physics. In the Appendix to our paper we show how partial commutativity relations, which played an important role in our definition of generalized quasi-symmetric functions, lead to Weyl commutation relations, a crucial ingredient of quantum physics.

APPENDIX A. PARTIAL COMMUTATIVITY CONDITIONS AND THE WEYL COMMUTATION RELATIONS

The result presented below is an output of Author's investigation [16, 15] of incidence-geometric description of bialgebra structure of the quantum plane, and was presented in Oberwolfach Conference *Discrete Differential Geometry* in July 2012.

Given $q \in \mathbb{k}^\times$, define the *quantum plane* $\mathbb{k}_q[x, y]$ as a quotient of the free algebra $\mathbb{k}\langle x, y \rangle$ in two variables by the Weyl commutation relation

$$(A.1) \quad yx = qxy.$$

This is a Noetherian domain and has therefore a division ring of fractions, denoted by $\mathbb{k}_q(x, y)$. When q is not a root of unity, the centre of the division ring is trivial, and hence $\mathbb{k}_q(x, y)$ is infinite dimensional over its centre [5].

The tensor product $\mathbb{k}_q[x, y] \otimes \mathbb{k}_q[x, y]$ can be realized as the quotient of the free algebra $\mathbb{k}\langle x_1, y_1, x_2, y_2 \rangle$ by the ultralocality (partial commutativity) conditions

$$(A.2) \quad x_1x_2 = x_2x_1, \quad y_1y_2 = y_2y_1, \quad x_1y_2 = y_2x_1, \quad y_1x_2 = x_2y_1,$$

supplemented by the Weyl commutation relations

$$(A.3) \quad y_1x_1 = qx_1y_1, \quad y_2x_2 = qx_2y_2.$$

Denote by $\mathbb{k}_q(x_1, y_1, x_2, y_2)$ its ring of fractions. By direct verification one can check the following result.

Proposition A.1. *Let us arrange the generators in 2×2 matrix A as follows*

$$(A.4) \quad A = \begin{pmatrix} y_1 & x_2 \\ x_1 & y_2 \end{pmatrix},$$

then A is invertible in $\mathbb{k}_q(x_1, y_1, x_2, y_2)$

$$(A.5) \quad A^{-1} = \begin{pmatrix} (y_1 - x_2y_2^{-1}x_1)^{-1} & (x_1 - y_2x_2^{-1}y_1)^{-1} \\ (x_2 - y_1x_1^{-1}y_2)^{-1} & (y_2 - x_1y_1^{-1}x_2)^{-1} \end{pmatrix} = \begin{pmatrix} \bar{y}_1 & \bar{x}_2 \\ \bar{x}_1 & \bar{y}_2 \end{pmatrix},$$

and the matrix elements of A^{-1} satisfy analogous ultralocality conditions

$$(A.6) \quad \bar{x}_1\bar{x}_2 = \bar{x}_2\bar{x}_1, \quad \bar{y}_1\bar{y}_2 = \bar{y}_2\bar{y}_1, \quad \bar{x}_1\bar{y}_2 = \bar{y}_2\bar{x}_1, \quad \bar{y}_1\bar{x}_2 = \bar{x}_2\bar{y}_1,$$

as well as the following Weyl commutation relations with q^{-1} in the place of q

$$(A.7) \quad \bar{y}_1 \bar{x}_1 = q^{-1} \bar{x}_1 \bar{y}_1, \quad \bar{y}_2 \bar{x}_2 = q^{-1} \bar{x}_2 \bar{y}_2.$$

We will show the opposite implication, which states that under some non-degeneracy conditions partial commutativity implies Weyl commutation relations. Let us start with four free symbols x_1, x_2, y_1, y_2 and the free division ring $F = F(x_1, x_2, y_1, y_2)$ over \mathbb{k} generated by the symbols (see [9] for necessary background). Arrange them into the matrix A as in (A.4). Then the matrix is invertible over $F(x_1, x_2, y_1, y_2)$, and its inverse can be written exactly in the form as given in equation (A.5). The free division rings $F_1 = F(x_1, y_1)$ and $F_2 = F(x_2, y_2)$ over \mathbb{k} , considered as subrings of F are separated, i.e. they intersect along the base field $F_1 \cap F_2 = \mathbb{k}$, which is their common centre.

Let \tilde{F} be a division ring [17] obtained from F by imposing partial commutativity conditions (A.2). Then F_1 and F_2 can be still considered as separated subrings of \tilde{F} . Centralizer of F_1 in \tilde{F} will be F_2 , and vice versa. By direct verification we have then the following result.

Proposition A.2. *If we impose partial commutation conditions (A.2) only, then elements of the matrix A^{-1} satisfy relations*

$$(A.8) \quad \bar{x}_1 \bar{y}_2 = \bar{y}_2 \bar{x}_1, \quad \bar{y}_1 \bar{x}_2 = \bar{x}_2 \bar{y}_1.$$

The following result closes our reasoning about in what sense partial commutativity of elements of the matrix A and its inverse A^{-1} implies Weyl commutation relations.

Theorem A.3. *Under assumptions of Proposition A.2, let elements of the inverse matrix A^{-1} given by (A.5) satisfy remaining partial commutativity conditions. Then there exists $q \in \mathbb{k}^\times$ such that (x_1, y_1) and (x_2, y_2) form Weyl pairs as given in (A.3).*

Proof. Under partial commutativity assumptions (A.2) we have

$$(A.9) \quad (\bar{x}_1 \bar{x}_2)^{-1} - (\bar{x}_2 \bar{x}_1)^{-1} = (y_2 - x_1 y_1^{-1} x_2) y_1 (y_1 x_1^{-1} x_2^{-1} y_2 - x_1^{-1} y_1 y_2 x_2^{-1}).$$

The first factor of RHS cannot vanish, because this would spoil invertibility of A , in particular we would have additional relation $x_1 x_2 = y_1 y_2$. By assumption of the Theorem expression (A.9) vanishes, therefore we have

$$(A.10) \quad x_1^{-1} y_1 x_1 y_1^{-1} = x_2 y_2^{-1} x_2^{-1} y_2,$$

which, by the separation assumption, implies that both sides must be equal to a constant $q \in \mathbb{k}^\times$ (vanishing of q would give presence of zero divisors) and leads directly to the statement (A.3). An identity

$$(A.11) \quad (\bar{y}_1 \bar{y}_2)^{-1} - (\bar{y}_2 \bar{y}_1)^{-1} = (y_1 - x_2 y_2^{-1} x_1) y_2 (x_1 y_1^{-1} y_2^{-1} x_2 - y_1^{-1} x_1 x_2 y_2^{-1}),$$

analogous to (A.9) gives the same result. □

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